UPWARD CLOSURE AND COHESIVE DEGREES[†]

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ABSTRACT

It is shown that the degrees of unsolvability of sets having almost any sort of immunity or cohesiveness property studied in recursion theory are closed upwards. From this it follows that every degree $\boldsymbol{a}$ with $\boldsymbol{a}' \geqq \mathbf{0}''$ contains a cohesive set.

A degree of unsolvability $\boldsymbol{b}$ is called *cohesive* if some cohesive set has degree $\boldsymbol{b}$. We show that if $\boldsymbol{b}$ is cohesive and $\boldsymbol{b} \leqq \boldsymbol{a}$, then $\boldsymbol{a}$ is cohesive. It follows from this by [2, Th. 4.1] that the condition $\boldsymbol{a}' \geqq \mathbf{0}''$ is sufficient for $\boldsymbol{a}$ to be cohesive. Actually the first-mentioned result will be proved in a very general form in which the property of cohesiveness (of sets) is replaced by an arbitrary property $\mathscr{P}$ of infinite sets which is hereditary under inclusion and possessed by at least one arithmetical set. Many examples of such properties, such as quasi-cohesiveness, have been studied in recursion theory.

We now give some definitions to facilitate the precise statement of the result alluded to above. A property $\mathscr{P}$ which holds only for infinite sets of numbers is called *hereditary under inclusion* if every infinite subset of a set having property $\mathscr{P}$ again has property $\mathscr{P}$. Any set having property $\mathscr{P}$ is called a *$\mathscr{P}$-set* and degrees represented by at least one $\mathscr{P}$-set are called *$\mathscr{P}$-degrees.* A class $\mathscr{C}$ of degrees is called *closed upwards* if whenever $\mathscr{C}$ contains a degree $\boldsymbol{b}$ it contains all degrees $\boldsymbol{a} \geqq \boldsymbol{b}$. Terminology used here without explanation is explained in [6].

Our main result is similar in form to [2, Th. 3.1] but has far weaker hypotheses.

THEOREM 1. *If $\mathscr{P}$ is any property of infinite sets which is hereditary under*

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inclusion and enjoyed by some arithmetical set, then the class of $\mathscr{P}$-degrees is closed upwards.

PROOF. Let us call a set B *rich* if every degree above that of B is represented by some subset of B.

LEMMA 1. *If B is infinite but not rich, then every arithmetical set is recursive in B.*

Before proving Lemma 1, we assume it and point out how the Theorem immediately follows. Let $\mathscr{P}$ satisfy the hypotheses of the Theorem and let $\boldsymbol{b}$ be a degree containing a $\mathscr{P}$-set B. To show that every degree $\geqq \boldsymbol{b}$ is a $\mathscr{P}$-degree it obviously suffices to show that there is a $\mathscr{P}$-set C which is rich and of degree $\leqq \boldsymbol{b}$. If B is rich, simply take C to be B. If B is not rich, let C be any arithmetical $\mathscr{P}$-set. By Lemma 1, C is recursive in B. Also it is an immediate consequence of the lemma that every infinite arithmetical set is rich, so C has all the desired properties.

The proof of Lemma 1 hinges on a result about recursive partitions from [3], so we repeat some definitions from that paper. If B is a set of numbers, let $[B]^k$ be the class of all k-element subsets of B. If $P \subseteq [N]^k$ (where N is the set of natural numbers), let $H(P)$ be the class of infinite sets B such that $[B]^k \subseteq P$ or $[B]^k \cap P = \varnothing$. Finally let $H^+(P)$ be the class of all sets B such that $B - D \in H(P)$ for some finite set D.

LEMMA 2. *If B is infinite and not rich and P is a recursive subset of $[N]^k$, then $B \in H^+(P)$.*

PROOF OF LEMMA 2. We assume that B is an infinite set not in $H^+(P)$ and prove that B is rich. Let A be a set in which B is recursive. We must show that B has a subset C of the same degree as A. We shall obtain C as $\bigcup_{i=1}^{\infty} C_i$ where for all i,

(i) $C_{i+1} \in [B]^k$

(ii) $\max(C_i) < \min(C_{i+1})$

(iii) $i \in A \leftrightarrow C_{i+1} \in P$.

The C_i's are defined by induction on i. Let C_0 be $\{0\}$. Given C_i, let C_{i+1} be the finite set of least index (in some fixed effective indexing) which satisfies (i)–(iii). To see that C_{i+1} exists, let $D_i = \{j : j \leqq \max C_i\}$. Since $B - D_i$ is infinite and not in $H(P)$, there are sets C_{i+1}^1, C_{i+1}^2 each satisfying (i), (ii) such that one but not the other belongs to P. Obviously one of these sets satisfies (iii). This completes the definition of the C_i's and thus of C, and it is easy to verify that C has the required properties.

Lemma 1 is now an immediate consequence of Lemma 2 above and of [3, Lemma 5.9]. (The latter implies that for all $k \geqq 1$ there is a recursive $P \subseteq [N]^{k+1}$ such that every element of $H(P)$ (and thus of $H^{+}(P)$) is of degree $\geqq \mathbf{0}^{k-1}$.)

COROLLARY 1. *If $\mathscr{P}$ is the class of cohesive sets, r-cohesive sets, quasi-cohesive sets, or infinite recursively indecomposable sets, or if $\mathscr{P}$ is $H(P)$ where P is a recursive subset of $[N]^k$ for some k, then the class of $\mathscr{P}$-degrees is closed upwards.*

COROLLARY 2. *If $\mathbf{a}' \geqq \mathbf{0}''$, then $\mathbf{a}$ is cohesive.*

Corollary 1 is an immediate consequence of the theorem, the existence of an arithmetical cohesive set, and of [3, Th. 5.5] (or the remark just after the proof of [3, Th. 5.3]). (Properties related to hyperhyperimmunity were not mentioned in Corollary 1 because these are already covered by [2, Th. 3.1].)

Corollary 2 follows from Corollary 1 and [2, Th. 4.1]. We close with some remarks and open questions.

1) Theorem 1 becomes false if its hypothesis is weakened by replacing "arithmetical" by "recursive in Kleene's $\mathcal{O}$". (To see this, let A be an infinite set recursive in $\mathcal{O}$ having no subset of higher degree [7, final paragraph] and let $\mathscr{P}$ be the class of infinite subsets of A.) We do not know whether Theorem 1 remains true if "arithmetical" is replaced by "hyperarithmetical". Similar comments apply to Lemma 1, in connection with which it can be shown from the final paragraph of [7] that only hyperarithmetical sets can be recursive in all infinite nonrich sets.

2) The proof of Theorem 1 is clearly not uniform. This nonuniformity is shared by the upward closure proofs in [2] and [4] but not by that of [5]. We do not know whether Theorem 1 or any of its applications in Corollary 1 have uniform proofs, although Corollary 2 can be proved uniformly by a more complicated argument.

3) Corollary 1 obviously remains true if the properties mentioned in it are relativized in the usual way to any fixed arithmetical set. We do not know what the situation is for relativization to an arbitrary set.

4) The converse to Corollary 2 is an apparently difficult open question.

Although we conjecture this converse to be false, S. B. Cooper has shown [1] that if $\boldsymbol{a} \leqq \mathbf{0}'$ and $\boldsymbol{a}$ is cohesive (or even hyperhyperimmune) then $\boldsymbol{a}' = \mathbf{0}''$.

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